The Sound of Symmetry

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Abstract. The inverse spectral problem was popularized by M. Kac's 1966 article in THIS MONTHLY "Can one hear the shape of a drum?" Although the answer has been known for over twenty years, many open problems remain. Intended for general audiences, readers are challenged to complete exercises throughout this interactive introduction to inverse spectral theory. The main techniques used in inverse spectral problems are collected and discussed, then used to prove that one *can* hear the shape of: parallelograms, acute trapezoids, and the regular n-gon. Finally, we show that one can *realistically* hear the shape of the regular n-gon amongst all convex n-gons because it is uniquely determined by a finite number of eigenvalues; the sound of symmetry can really be heard!

1. INTRODUCTION. Have you heard the question, "Can one hear the shape of a drum?" Do you know the answer? This question is the title of an article published in 1966 by M. Kac [**21**] based on the following.

Question 1. *If two planar domains have the same spectrum, are they identical up to rigid motions of the plane?*

For a bounded domain Ω in $\mathbb{R}^2$, the above mentioned *spectrum* is the set of eigenvalues of the Laplace operator Δ with Dirichlet boundary condition. This is the set of all real numbers λ such that there exists a function $u \neq 0$ which is $\mathcal{C}^2$ on the interior of the domain, continuous up to the boundary, and satisfies both the Laplace equation and the boundary condition:

$$\Delta u(x, y) := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda u, \quad u = 0 \text{ on the boundary of } \Omega. \tag{1.1}$$

In the language of functional analysis, the Laplace operator with Dirichlet boundary condition is an essentially self-adjoint unbounded operator densely defined on the Hilbert space $\mathcal{L}^2(\Omega)$; details are in [**9**]. For our purposes readers can be assuaged that prerequisite knowledge of functional analysis is unnecessary.

The eigenvalues form a discrete subset of $(0, \infty)$, $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, which is in bijection with the resonant frequencies a drum would produce if Ω were its drumhead. With a perfect ear one could hear all these frequencies and therefore know the spectrum. Based on this physical description, Kac paraphrased Question 1 as, "Can one hear the shape of a drum?" In other words, if two drums sound identical to a perfect ear and therefore have identical spectra, then are they the same shape?

Hearing a string. Consider the simplest case, that our drum is actually a string, for example, on a guitar or a violin. The vibration caused by plucking a string of length ℓ and keeping the ends fixed is mathematically described by the wave equation

$$\frac{\partial^2 F}{\partial x^2} = c^2 \frac{\partial^2 F}{\partial t^2}.$$

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Separating variables by assuming F can be expressed as $F(x, t) = f(x)g(t)$, and scaling the time variable so that $c = 1$ leads to the equation $f''(x)g(t) = f(x)g''(t)$. Dividing both sides by $f(x)g(t)$, we have

$$\frac{f''(x)}{f(x)} = \frac{g''(t)}{g(t)}$$

which implies both sides are equal to a constant. Since the ends of the string are fixed, $f(0) = f(\ell) = 0$. By calculus the solutions are

$$f_k(x) = \sin\left(\frac{k\pi x}{\ell}\right), \quad \text{with } \lambda = \lambda_k = \frac{k^2\pi^2}{\ell^2}, \quad k \in \mathbb{N}.$$

Question 2. *Can one hear the shape of a string?*

The "shape" of the string is just its length, so we can formulate the question as: if we know the set of all $\{\lambda_k\}_{k=1}^{\infty}$, then do we know the length of the string? The answer is yes, and we actually only need to know λ_1, because $\ell = \sqrt{\frac{\pi^2}{\lambda_1}}$.

This shows that we can hear the length of the string based on the first eigenvalue. Musically inclined readers already know this because λ_1 determines the fundamental tone of the string, and overtones corresponding to λ_k for $k \geq 2$ have frequencies which are integer multiples of λ_1. This results in a pleasant sound.

What happens if our instrument is played by the vibration of a membrane or higher dimensional object? In the process of solving Exercise 1 below, readers will see that the overtones of higher dimensional "instruments" are not necessarily integer multiples of the fundamental tone. Musically such instruments may produce "terrible sounds" as observed by C. Gordon in [**16**]. Some of these terrible sounds can be heard online at S. M. Belcastro's website http://www.toroidalsnark.net/som.html which uses a program written by D. DeTurck.

The answer to Kac's question, open problems, and our results. Now that we have answered the question in one dimension, let us return to Kac's question in two dimensions. Although perhaps the most natural drum is a circular drum, rectangular drums are a bit easier to handle. Translating, rotating, or reflecting the domain does not change the numbers λ in (1.1), so we can assume that the domain has vertices $(0, 0)$, $(\ell, 0)$, $(0, w)$, (ℓ, w).

Exercise 1. *Prove that one* can *hear the shape of a rectangle.*

Unfortunately it is only possible to compute the eigenvalues in closed form for a few special examples, such as rectangles and disks. Without expressions for the eigenvalues, how can we answer Kac's question? Mathematically the question is equivalent to determining whether or not the following map

$$\Lambda : \mathcal{M} \to \mathbb{R}^{\infty}, \quad \Omega \mapsto (\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots),$$

is injective, where $\mathcal{M}$ is the moduli space of all bounded domains with piecewise smooth boundary in $\mathbb{R}^2$. This moduli space is simply the set of equivalence classes of domains modulo rigid motions of the plane.

Kac's article appeared in print two years after a lovely one-page paper by Milnor [**24**] which showed that *one cannot hear the shape of a 16 dimensional drum.* Milnor used a construction of Witt [**30**] of self-dual lattices in $\mathbb{R}^{16}$ which are distinct in the sense that no rigid motion of $\mathbb{R}^{16}$ exists which maps one lattice to the other. Consequently, Milnor was able to give a short proof that the corresponding tori are not isometric, but have the same set of eigenvalues.

In 1985, Sunada introduced what he described as "a geometric analogue of a routine method in number theory," which became known as *the Sunada method* [**28**] and can be used to produce large numbers of isospectral manifolds in four dimensions which are not isometric. Buser generalized this method to construct isospectral nonisometric surfaces [**5**]. However, this was still not quite the answer to Kac's question, since curved surfaces are not planar domains.

Gordon, Webb, and Wolpert saw nonetheless that the basic ideas of Buser could be used to prove that the answer to Kac's question is "no," by showing that the map Λ is not injective on $\mathcal{M}$ [**17**], [**18**]. They proved that the two domains in Figure 1 would sound identical to a perfect ear because they are isospectral, but as can be seen, the domains are not identical up to rigid motions. Chapman published a charming article which shows how to construct isospectral nonisometric planar domains by folding paper [**7**].

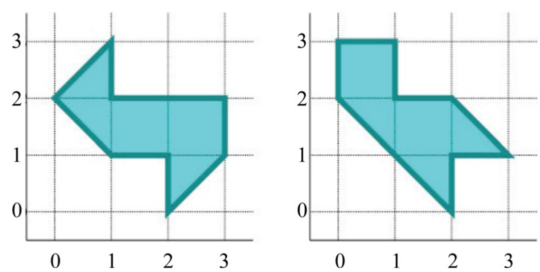

Figure 1. Identical sounding drums [**17**].

Although it may seem that Kac's question was laid to rest in the 1990s, many open problems remain. If we cannot determine the shape of the domain completely by its spectrum, can we at least determine, or *hear*, some of its geometric features?

Motivated by these problems we shall investigate the injectivity of Λ restricted to certain subsets of $\mathcal{M}$. A natural choice is the moduli space of convex n-gons, $\mathfrak{M}_n$, the set of all convex n-gons in the plane modulo equivalence under rigid motions. A convex n-gon is uniquely determined by $2n - 3$ real parameters, hence the moduli space of convex n-gons can be identified with a $2n - 3$ dimensional orbifold. The precise geometry of $\mathfrak{M}_n$, albeit of independent interest, does not play a significant role for our purposes.

Surprisingly, even within the moduli space of convex n-gons Kac's question is a subtle problem. Durso proved [**12**] that one *can* hear the shape of a triangle, so in fact Λ restricted to the moduli space of Euclidean triangles is injective (see also [**19**]). For n-gons with $n > 3$, the injectivity of Λ restricted to the $\mathfrak{M}_n$ is a widely open problem.

In this note we present three theorems with simple proofs. To the best of our knowledge, our results are new but more importantly, their proofs show most of the basic methods of inverse spectral theory in an elementary way.

Our first result extends Durso's theorem to parallelograms and acute trapezoids (see Definition 3), showing that one can hear the shape of parallelograms and acute trapezoids.

Theorem 3. *If two parallelograms have the same spectrum, then they are congruent. If two acute trapezoids have the same spectrum, then they are congruent.*

Our next result shows that one can also hear symmetry in the following sense.

Theorem 4. *If an n-gon is isospectral to a regular n-gon, then they are congruent.*

Theorem 4 shows that the symmetry of the regular n-gon can be heard among all n-gons, which in the spirit of Kac we paraphrase as follows.

Among all n-gons, one can hear the symmetry of the regular one.

Theorems 3 and 4, like Durso's theorem, use the *entire* spectrum. Physically this is like having a *perfect* ear, which is impossible. It is therefore interesting to consider isospectral problems involving a finite part of the spectrum. A well known conjecture due to Pólya and Szegő is the following.

Conjecture 1 (Pólya–Szegő). *For each $n \geq 5$, the regular n-gon uniquely minimizes λ_1 among all n-gons with fixed area. For $n = 3, 4$, this is a theorem proven by Pólya and Szegő in [**26**].*

We are presently unable to prove the conjecture, however our last result may be seen as a weaker statement, and so we refer to this as the weak Pólya–Szegő theorem.

Theorem 5 (Weak Pólya–Szegő). *For each $n \geq 3$ there exists N which depends only on n such that if the first N eigenvalues of a convex n-gon coincide with those of a regular n-gon, then it is congruent to that regular n-gon.*

We begin in § 2 with an overview of methods used in inverse spectral problems. These are then used to prove Theorem 3 in § 3 and Theorem 4 as well as the weak Pólya–Szegő theorem (Theorem 5) in § 4. Conjectures and open problems comprise § 5.

2. METHODS. Some readers may be familiar with the "Steiner symmetrization" technique, named after the Swiss mathematician Jakob Steiner [**27**]. Mathematical objects are often separated into three classes. For example, conic sections can be classified as parabolic, elliptic, or hyperbolic. Steiner would have recognized this as an example of an old German saying "Alle gute Dinge sind drei." (All good things come in three.) In this tradition we have distinguished three general types of methods used in spectral theory. Unlike conic sections, however, these methods have a nonempty intersection.

Geometric techniques. Steiner symmetrization is an example of a more general technique known in this context as *local adjustment*.

Quantities which are determined by the spectrum are known as *spectral invariants,* so one can say that these quantities can be "heard." We shall see in § 2 that the area and the perimeter of a domain can be heard. Consequently it is possible to "hear" the following quantity:

$$f(\Omega) := \frac{|\Omega|}{|\partial\Omega|^2}, \tag{2.1}$$

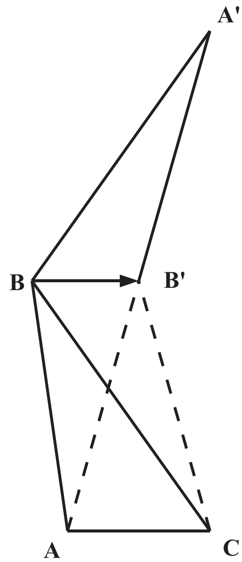

Figure 2. Local adjustment to equal sides increases f.

where $|\Omega|$ denotes the area of a domain Ω, and $|\partial\Omega|$ denotes its perimeter. Note that f is invariant under scaling of the domain. Consequently, if two domains Ω and Ω' are equivalent up to translation, reflection, rotation, and scaling by a constant factor, then $f(\Omega) = f(\Omega')$.

The ancient Greeks (see [**4**]) proved that for any polygon Ω, $f(\Omega) \leq f(\mathbb{D})$, known as the *isoperimetric inequality,* where $\mathbb{D}$ is a disk. Steiner also gave a proof of this fact and generalized the result to three dimensions in [**27**]. He began by stating the following "fundamental theorem."

[Steiner 1838] *Among all triangles with the same base and height, the isosceles has the smallest perimeter.*

We include a short proof by picture. Consider Figure 2 in which the sides of the triangle ABC satisfy $|BC| > |AB|$. Moving the vertex B parallel to the base segment AC creates a new vertex B' such that the side lengths $|AB'| = |CB'|$. To compare the new triangle to the original triangle, we reflect the triangle CBB' across the segment BB'. This creates a new point A' which extends the segment AB'. By symmetry the lengths $|A'B'| = |B'C| = |AB'|$, and $|A'B| = |BC|$. Since A, B', and A' are collinear, the distance from A to A', $|AA'| = |AB'| + |B'A'| = |AB'| + |B'C|$ is less than the distance from A to B and then to A', $|AB| + |BA'|$. Now, since $|BA'| = |BC|$,

$$|AB'| + |AC| + |B'C| < |AB| + |AC| + |BA'| = |AB| + |AC| + |BC|.$$

Since the triangles ABC and $AB'C$ have the same height and base, they have the same area. The above inequality shows that the triangle $AB'C$ has strictly smaller perimeter than that of triangle ABC. Consequently, local adjustment to equal sides increases the value of f.

Steiner went on to prove that an arbitrary convex polygon can be deformed to a convex polygon with the same area, smaller perimeter, and which is symmetric about an axis, a technique now known as "Steiner symmetrization." A similar technique was used by Polyá and Szegő to prove Conjecture 1 for $n \leq 4$. The proof no longer works

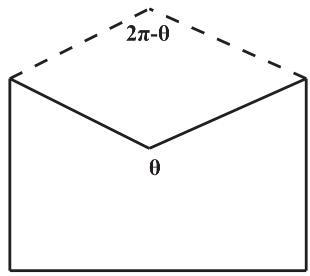

Figure 3. Reflecting a concave vertex to a convex vertex increases area and preserves perimeter.

for $n \geq 5$ because Steiner's symmetrization generally increases the number of sides as can be seen in the illustrations of [**27**]. We shall prove a variation of Steiner's isoperimetric inequality using adjustment techniques which preserve the number of sides.

Proposition 6. *The function f defined in* (2.1) *is uniquely maximized among all n-gons by the regular n-gons.*

Proof. We first note that reflecting across a concave vertex is similar to opening an envelope (see Figure 3) and preserves the perimeter while increasing the area thereby increasing f, so we only need to consider convex n-gons.

We denote by R_n a regular n-gon and note that $f(R_n) = \frac{1}{4n\tan(\pi/n)}$ is a monotonically increasing sequence which converges to $f(\mathbb{D})$ as $n \to \infty$. Since f is bounded above on $\mathfrak{M}_n$ by $f(\mathbb{D})$, the supremum of f on $\mathfrak{M}_n$, denoted by f_∞, is the limit of $f(P_k)$ as $k \to \infty$ for a sequence of convex n-gons $\{P_k\}$. Since $f_\infty \geq f(R_n)$ no subsequence of P_k can collapse to a segment. Moreover, since f is invariant under scaling, we may assume that P_k has a diameter equal to one for each k, and consequently the side lengths are contained in $(0, 1]$, and the angles are contained in $(0, \pi)$. By passing to a subsequence, if necessary, we may assume that $P_k \to P$ as $k \to \infty$, where P is an m-gon for $m \leq n$. We proceed by induction on n. In the base case $n = 3$, and since P_k cannot collapse to a segment, P is a triangle. By Steiner's fundamental theorem P must be equilateral (see Figure 2); the details of this proof are left to the reader.

We now assume that the proposition is proven for any j such that $3 \leq j < n$. Since area and perimeter are continuous functions of the domains, $P_k \to P$ implies $f(P_k) \to f(P) = f_\infty$. We claim that P must also be an n-gon, for if P is an m-gon for $m < n$, then we have

$$f(P) \leq f(R_m) < f(R_n) \leq f_\infty = f(P),$$

which is a contradiction. By Steiner's fundamental theorem P is equilateral.

As soon as $n > 3$, equilateral no longer implies equiangular, so we require an additional local adjustment argument. The proof may seem a bit long, but the idea is simple: we consider a continuous variation of P with respect to a parameter t with $t = 0$ corresponding to P. Then, since P maximizes f, we must have $f'(0) = 0$. Since f is scale invariant, and P has equal sides, we shall assume that the sides of P all have length equal to one. Consider the edge between two vertices v_i and v_{i+1}, where the vertices are considered modulo n. Denote by $\{\gamma_i\}_{i=1}^n$ the set of interior angles of P, and the exterior angles $\alpha_i := \pi - \gamma_i$. Consider moving the edge between v_i and v_{i+1} in the direction of the outward normal to this edge as depicted in Figure 4. Let t denote the distance the edge is translated. Then the area of this

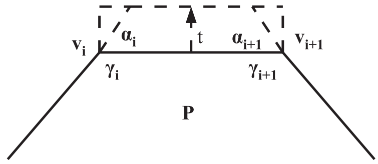

Figure 4. Local adjustment to equal angles increases f.

deformation is $A(t) = A + t + O(t^2)$, where A is the area of P. The perimeter $L(t) = L + t\csc\alpha_i - t\cot\alpha_i + t\csc\alpha_{i+1} - t\cot\alpha_{i+1} + O(t^2)$, where L is the perimeter of P. Let

$$\varphi(x) := \csc x - \cot x = \tan\left(\frac{x}{2}\right).$$

The function f can be considered as a function of the parameter t, where $t = 0$ corresponds to P, and since P maximizes the function f, $f'(0) = f'(P) = 0$. Substituting the expressions for $A(t)$ and $L(t)$ above,

$$f(t) = \frac{A(t)}{L(t)^2} = \frac{A + t + O(t^2)}{(L + t(\varphi(\alpha_i) + \varphi(\alpha_{i+1})) + O(t^2))^2}.$$

Therefore,

$$f'(0) = \frac{1}{L^2} - \frac{2A}{L^3}(\varphi(\alpha_i) + \varphi(\alpha_{i+1})) = 0 \implies \varphi(\alpha_i) + \varphi(\alpha_{i+1}) = \frac{L}{2A}.$$

Since $1 \le i \le n$ was chosen arbitrarily, $\varphi(\alpha_i) + \varphi(\alpha_{i+1}) = \frac{L}{2A}$ for all i modulo n shows that similarly $\varphi(\alpha_{i+2}) + \varphi(\alpha_{i+1}) = \frac{L}{2A}$ for all i modulo n. Since $\varphi(x) = \tan(x/2)$ is a monotonically increasing function on $(0, \pi)$, and the exterior angles $\alpha_i \in (0, \pi)$, φ is injective, and therefore $\alpha_i = \alpha_{i+2}$ for all i modulo n. If n is odd, the proof is complete because

$$\alpha_1 = \alpha_3 = \cdots = \alpha_n = \alpha_{n+2} = \alpha_2 = \alpha_4 = \cdots = \alpha_{n-1},$$

which shows that P is equiangular. If n is even,

$$\sum_{i=1}^{n} \gamma_i = \pi(n-2), \text{ and } \alpha_i = \pi - \gamma_i, \ \forall i = 1, \ldots, n \implies \sum_{i=1}^{n} \alpha_i = 2\pi.$$

Since $\alpha_i = \alpha_{i+2}$ for all i modulo n, $\frac{n}{2}\alpha_1 + \frac{n}{2}\alpha_2 = 2\pi$, so $\alpha_1 + \alpha_2 = \frac{4\pi}{n}$. Consequently

$$\varphi(\alpha_1) + \varphi(\alpha_2) = \varphi(\alpha_1) + \varphi\left(\frac{4\pi}{n} - \alpha_1\right) = \tan\left(\frac{\alpha_1}{2}\right) + \tan\left(\frac{2\pi}{n} - \frac{\alpha_1}{2}\right).$$

Since the sides all have length 1, $L = n$. Therefore, we have

$$\varphi(\alpha_1) + \varphi(\alpha_2) = \tan\left(\frac{\alpha_1}{2}\right) + \tan\left(\frac{2\pi}{n} - \frac{\alpha_1}{2}\right) = \frac{1}{2n} \cdot \frac{1}{f(P)} = \frac{n}{2A}.$$

Since P minimizes $1/f$, a straightforward computation shows that

$$\alpha_1 = \frac{2\pi}{n} \implies \alpha_2 = \frac{4\pi}{n} - \alpha_1 = \frac{2\pi}{n} \implies \alpha_i = \frac{2\pi}{n} \quad \forall i. \qquad \blacksquare$$

Analytic techniques. The spectrum obeys the following scaling property:

$$\Omega \mapsto c\Omega \implies \lambda_k(\Omega) \mapsto c^{-2}\lambda_k(\Omega). \tag{2.2}$$

Musically this corresponds to the fact that longer strings make lower tones, whereas shorter strings make higher tones. More generally, the bigger the drum, the lower the tones. This fact is known as domain monotonicity and is given in (2.5) below.

Variational principles. The eigenvalues can be defined by a *variational principle,* also known as a "mini-max" principle, given by the infima of the Rayleigh–Ritz quotient

$$\lambda_k = \inf_{f \in H_0^{1,2}(\Omega),\, f \not\equiv 0,\, f \perp f_j,\, j=0,\dots,k-1} \left\{ \frac{\int_\Omega |\nabla f|^2 dxdy}{\int_\Omega f^2 dxdy} \right\}. \tag{2.3}$$

Above $f_0 \equiv 0$, f_j is an eigenfunction for λ_j for $j \geq 1$, and orthogonality is with respect to $\mathcal{L}^2(\Omega)$. This formula can be found in [**8**]. An equivalent formula found in [**9**] is the so-called maxi-min principle

$$\lambda_k = \inf_{L \subset H_0^{1,2}(\Omega),\, \dim(L)=k} \left\{ \sup_{f \in L} \frac{\int_\Omega |\nabla f|^2 dxdy}{\int_\Omega f^2 dxdy} \right\}. \tag{2.4}$$

The variational principles can be used to show that the eigenvalues are continuous functions of the domains, so if a sequence of domains $\Omega_k \to \Omega_0$ as $k \to \infty$, then the eigenvalues

$$\lambda_i(\Omega_k) \to \lambda_i(\Omega_0), \quad \text{as } k \to \infty, \text{ for each } i \in \mathbb{N}.$$

The maxi-min principle also implies *domain monotonicity:*

$$\Omega \subseteq \Omega' \implies \lambda_i(\Omega) \geq \lambda_i(\Omega'), \quad \forall i. \tag{2.5}$$

Fundamental gap. The difference between the first two eigenvalues is known as the *fundamental gap.* A theorem proven by Andrews and Clutterbuck [**1**] shows that if the diameter of a convex domain tends to zero, then its fundamental gap blows up. On the other hand, we will see in the proposition below that if the diameter of a convex domain tends to infinity, then its fundamental gap tends to 0.

Recall that the *in-radius* of a domain is the supremum over all radii of disks which can be inscribed in the domain. Since disks will play an important role in the proof of Proposition 7 below, the reader is invited to contribute to that proof by completing the following.

Exercise 2. *Let $j_{m,n}$ be the mth zero of the Bessel function J_n of order n. Prove that the eigenvalues of the disk of radius R are $\left(\frac{j_{m,n}}{R}\right)^2$ for $m \geq 1$ and $n \geq 0$. In particular, the first eigenvalue is $\left(\frac{j_{1,0}}{R}\right)^2$.*

Proposition 7. *If a sequence of convex domains $\{\Omega_k\}_{k=1}^\infty$ in $\mathbb{R}^2$ satisfies*

$$\lambda_i(\Omega_k) = \lambda_i(\Omega_1), \quad \forall k \in \mathbb{N}, \quad i = 1, 2,$$

then both the diameters and the in-radii of the domains are contained in a compact subset of $(0, \infty)$.

Proof. If the in-radii of the domains $r_k \to \infty$, then the domains contain increasingly larger disks, so we can estimate using domain monotonicity. By Exercise 2 and domain monotonicity, $\lambda_1(\Omega_1) = \lambda_1(\Omega_k) \leq \frac{1}{r_k^2} j_{1,0}^2$ which tends to 0 as $r_k \to \infty$, a contradiction.

On the other hand, if the in-radii tend to 0, then there are rectangles R_k of height h_k such that $\Omega_k \subset R_k$ and $h_k \to 0$. By domain monotonicity again, $\lambda_1(\Omega_1) = \lambda_1(\Omega_k) \geq \lambda_1(R_k) > \frac{\pi^2}{h_k^2}$ which tends to ∞ as $k \to \infty$, a contradiction. Since the diameter is bounded below by the in-radius, the diameters also cannot tend to 0.

Finally we show that assuming the in-radii neither collapse nor explode while the diameters tend to ∞ leads to a contradiction. Define the *width* w_k of Ω_k to be the shortest distance between two infinite parallel lines such that Ω_k fits within a strip of width w_k. The in-radii are uniformly bounded away from both 0 and ∞ and so the widths are also bounded uniformly away from 0 and ∞. Next, we rotate and translate the domains such that the width w_k goes from the point $A := (0, w_k/2)$ to the point $B := (0, -w_k/2)$, and these points lie on the boundary of Ω_k. By domain monotonicity, since Ω_k is contained in a rectangle of width w_k,

$$\lambda_1(\Omega_k) \geq \lambda_1(\text{Rectangle of width } w_k) > \frac{\pi^2}{w_k^2}.$$

Reflecting across the vertical axis if necessary, we may assume there is a point in Ω_k whose horizontal coordinate is $d_k/3$. We will call this point C; see Figure 5.

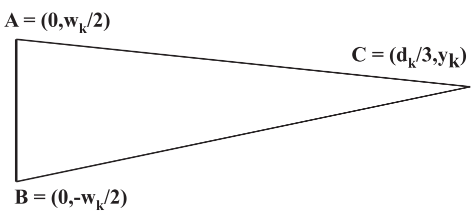

Figure 5. Triangle T_k.

By convexity, triangle $T_k := ABC$ is contained in Ω_k. It follows from the proof of Proposition 1 of [**22**] (see also similar estimates in [**14**], [**15**]) that

$$\lambda_2(T_k) \leq \frac{\pi^2}{w_k^2} + c d_k^{-2/3} \tag{2.6}$$

for a fixed constant $c > 0$. Consequently, by domain monotonicity since $T_k \subset \Omega_k$,

$$\lambda_2(\Omega_k) \leq \lambda_2(T_k) \leq \frac{\pi^2}{w_k^2} + c d_k^{-2/3}.$$

Together with the lower bound for $\lambda_1(\Omega_k)$ we have the following contradiction: $\lambda_2(\Omega_1) - \lambda_2(\Omega_1) = \lambda_2(\Omega_k) - \lambda_1(\Omega_k) \leq c d_k^{-2/3} \to 0$ as $d_k \to \infty$. This shows that the diameters cannot diverge and hence both the diameters and the in-radii must be contained in a compact subset of $(0, \infty)$. ∎

Exercise 3. *Prove* (2.6) *by showing that the triangle in Figure 5 contains a rectangle of height* $w_k(1 - d_k^{-2/3})$ *and width* $\frac{1}{3}d_k^{1/3}$ *and then use the domain monotonicity.*

The heat trace. The spectrum not only determines the resonant frequencies of vibration, but also the flow of heat. The heat trace is

$$\sum_{k=1}^{\infty} e^{-\lambda_k t}.$$

The eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ grow at an asymptotic rate known as *Weyl's law* [**29**] which can be used to prove that the above sum converges uniformly for all $t \geq t_0 > 0$ for any positive t_0 but diverges as $t \downarrow 0$. Weyl's law can also be used to show that the way in which the heat trace diverges as $t \downarrow 0$ determines certain geometric features such as the area. Pleijel [**25**] proved that the heat trace admits an asymptotic expansion as $t \to 0$ of the form

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}}, \qquad t \downarrow 0,$$

where $|\Omega|$ denotes the area of Ω, and $|\partial\Omega|$ is the perimeter. Kac determined the third term in the asymptotic expansion of the heat trace for bounded domains in [**21**], and we briefly recall the key ideas from those calculations.

The heat kernel on $\mathbb{R}^2$ can be explicitly computed to be

$$H_E(x, y, x', y', t) = \frac{1}{4\pi t} e^{-|(x,y)-(x',y')|^2/4t}.$$

Integrating along the diagonal $x = x'$, $y = y'$ over a bounded region $R \subset \mathbb{R}^2$ gives

$$\int_R \frac{1}{4\pi t} dx dy = \frac{|R|}{4\pi t}.$$

At each interior point of the domain Ω there is a neighborhood which does not intersect the boundary, and on this neighborhood the heat kernel for Ω is "close" to the Euclidean heat kernel H_E for short times. Kac referred to this as "not feeling the boundary" [**21**]. He showed that the heat trace for a planar domain is asymptotic to the trace over the domain of the Euclidean heat kernel,

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \int_\Omega H_E(x, y, x, y, t) dx dy = \frac{|\Omega|}{4\pi t}, \quad t \downarrow 0.$$

To compute the next term in the asymptotic expansion, he considered the behavior at the boundary both near and away from the corners. The principle of not feeling the boundary is what we call a "locality principle," which means that if one cuts a domain into neighborhoods and uses a model heat kernel for each neighborhood (with the correct boundary condition if the neighborhood intersects the boundary), then the heat kernel for the domain Ω integrated over each neighborhood is asymptotically equal as $t \downarrow 0$ to the model heat kernel integrated over the same neighborhood. More precisely, the heat trace for a locally constructed parametrix is asymptotically equal to the actual heat trace over Ω as $t \downarrow 0$. In the case of a polygonal domain there are three types of neighborhoods and three model heat kernels: the Euclidean heat kernel for interior neighborhoods, the heat kernel for a half plane for edge neighborhoods away from the vertices, and the heat kernel for an infinite circular sector of opening angle equal to that of the opening angle at a vertex. Kac used the explicit formulas for these model heat kernels and the locality principle to show[1] that for a convex n-gon Ω with interior angles $\{\alpha_i\}_{i=1}^n$,

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{i=1}^{n} \frac{\pi^2 - \alpha_i^2}{24\pi\alpha_i} \quad t \downarrow 0. \tag{2.7}$$

This shows that the area, the perimeter, and the sum over the angles of $\frac{\pi^2-\alpha^2}{24\pi\alpha}$ are all spectral invariants.

The wave trace. The spectrum also determines the *wave trace.* Imagine a convex n-gon is a billiard table. The set of closed geodesics is precisely the set of all paths along which a billiard ball would roll such that it eventually returns to its starting point and then continues rolling along exactly the same path again. The set of lengths of closed geodesics, which is known as *the length spectrum,* is related to the (Laplace) spectrum by a deep result proven by Duistermaat and Guillemin [**11**] in the late 1970s. The wave trace is a tempered distribution defined by

$$\sum_{k=1}^{\infty} e^{i\sqrt{\lambda_k}t}.$$

Duistermaat and Guillemin proved in [**11**] that the singularities of the wave trace are contained in the length spectrum. If we know the entire spectrum $\{\lambda_k\}_{k=1}^{\infty}$, then we know the wave trace and therefore the times at which it is singular, so these times are spectral invariants. It is an open problem to determine whether the singularities of the wave trace are *properly* contained in the length spectrum. Under a certain "clean intersection hypothesis" given in [**11**], the singularities of the wave trace coincide precisely with the length spectrum, and many people expect that this hypothesis is *always* satisfied. For our purposes, Hillairet proved in [**20**]*Theorem 2, that for an n-gon, the length of the shortest closed geodesic is one of the singularities of the wave trace, and the length of this geodesic is uniquely determined by the wave trace. Consequently, the length of the shortest closed geodesic in an n-gon is a spectral invariant. As we shall see in § 3, it can be significantly more complicated to extract geometric information from the wave trace than it is to extract information from the heat trace. This is one reason the wave trace is often considered a more subtle spectral invariant than the heat trace.

Algebraic techniques. Shortly after Durso completed her Ph.D. thesis [**12**], Chang and DeTurck published the following isospectrality result for triangular domains.

Theorem 8 (Chang and DeTurck [6]). *Let T_0 be a Euclidean triangle. There is an integer N which depends only on the first two eigenvalues of T_0 such that if T_1 is another triangle whose first N eigenvalues coincide with those of T_0, then all the eigenvalues coincide.*

[1] Kac did not compute the closed formula we have here, which is due to Dan Ray (unpublished) and Fedosov (in Russian) [**13**] and appeared in [**23**]; a particularly transparent proof is in [**3**].

Triangular domains can be parametrized to depend analytically on three parameters. The above theorem is an application of the following more general result for families of Riemannian metrics depending analytically on finitely many parameters.

Theorem 9 (Chang and DeTurck [6]). *Let D be a compact oriented manifold (with or without boundary) of dimension n. We consider a family of metrics $g(\varepsilon)$, depending analytically on the parameter $\varepsilon \in \mathbb{R}^p$. Let $\lambda_k(\varepsilon)$ denote the kth eigenvalue of the Laplacian of $g(\varepsilon)$ on D, and if $\partial D \neq \emptyset$ we assume that ∂D is piecewise smooth, and we impose the Dirichlet boundary condition. We also let $\sigma(\varepsilon)$ denote the spectrum of the Laplacian of $g(\varepsilon)$. Under these assumptions, for each compact subset $K \subset \mathbb{R}^p$ there is an integer $N = N(K)$ such that if $\varepsilon_0, \varepsilon_1 \in K$ and $\lambda_j(\varepsilon_0) = \lambda_j(\varepsilon_1)$ for all $j = 1, \ldots, N$, then $\sigma(\varepsilon_0) = \sigma(\varepsilon_1)$. In other words, for $\varepsilon \in K$ the entire spectrum of the Laplacian of $g(\varepsilon)$ is determined by the values of the first $N(K)$ eigenvalues.*

From [**6**] we quote, "The key ingredients in the proof are the assumption of real-analytic dependence of the metric on the parameters, and the resulting real-analytic dependence of certain symmetric functions of the eigenvalues, and finally the fact that the ring of germs of real analytic functions of finitely many variables is Noetherian." To use the above result, we require the following.

Proposition 10. *For each $n \geq 3$, $\mathfrak{M}_n$ can be identified with a family of Riemannian metrics on the unit disk which depend real analytically on $(2n-3)$ parameters.*

Proof. Let $\mathbb{D}$ denote the unit disk. The uniformization theorem states that all simply connected bounded domains in the plane are conformally equivalent to the disk. For polygonal domains, there is an explicit formula for the conformal map known as the Schwarz–Christoffel formula. Let

$$f(z) := \int_0^z \prod_{i=1}^n (w - w_i)^{\alpha_i - \pi} dw : z \in \mathbb{D} \to f(z) \in P. \tag{2.8}$$

This function is a conformal map from the disk $\mathbb{D}$ to the polygon P with interior angles $\{\alpha_i\}_{i=1}^n$ and vertices $p_i = f(w_i)$, where the points w_i lie on the boundary of the unit disk. Fix points w_1 and w_2 in $\partial\mathbb{D}$, and assume that the length of the shortest side of P is $|p_1 - p_2| = |f(w_1) - f(w_2)| = 1$. By Theorem 3.1 of [**10**], the $n-1$ angles $\{\alpha_i\}_{i=1}^{n-1}$ together with the $n-3$ side lengths

$$|p_i - p_{i+1}|, \quad i = 2, \ldots, n-1,$$

uniquely determine P under the assumption that the shortest side length of P is equal to one. Moreover, since the points p_1 and p_2 and the angles are fixed, the side lengths $|p_i - p_{i+1}|$ uniquely determine the location of the points $w_3, \ldots, w_n \in \partial\mathbb{D}$. We therefore define

$$f_\varepsilon(z) := cf(z), \quad \varepsilon := (\alpha_1, \ldots, \alpha_{n-1}, p_3, \ldots, p_n, c) \in \mathbb{R}^{2n-3}.$$

The constant c above relaxes the assumption that the length of the shortest side of the polygon is 1. Defined as such, this function is holomorphic in $\mathbb{D}$, piecewise holomorphic on $\partial\mathbb{D}$, and continuous up to $\partial\mathbb{D}$. The metrics $g(\varepsilon)$ on $\mathbb{D}$, where $g(\varepsilon)$ is the pull-back of the Euclidean metric on P with respect to the function f_ε, vary analytically with respect to the parameter $\varepsilon \in \mathbb{R}^{2n-3}$. The spectrum of the Euclidean Laplacian on

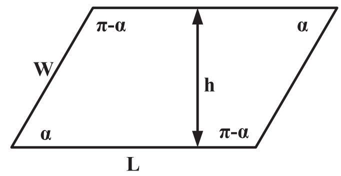

Figure 6. A parallelogram.

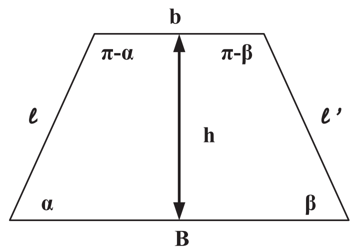

Figure 7. An acute trapezoid.

P with Dirichlet boundary condition is identical to the spectrum of the Laplacian with respect to the metric $g(\varepsilon)$ on $\mathbb{D}$ with Dirichlet boundary condition. ■

3. HEARING QUADRILATERALS. With the techniques introduced in the last section, we shall first prove that one can hear the shape of a parallelogram.

Proof of hearing a parallelogram. Let the length of the longest side of the parallelogram be denoted by L, and the length of the adjacent side be denoted by W. The perimeter $P = 2(L + W)$. If the height of the parallelogram is h, then the area is $A = Lh$. The parallelogram has four interior angles, the smallest two measure $\alpha \leq \frac{\pi}{2}$. The other two angles each have measure $\pi - \alpha$. So the constant term in the short time asymptotic expansion of the heat trace (2.7) is $a_0 = \frac{\pi^2}{12\alpha(\pi-\alpha)} - \frac{1}{12}$. Consider the function $f(\alpha) = \frac{1}{\alpha(\pi-\alpha)}$, and its derivative

$$f'(\alpha) = \frac{2\alpha - \pi}{(\pi\alpha - \alpha^2)^2} = 0 \iff \alpha = \frac{\pi}{2}.$$

The function f is strictly decreasing on $(0, \pi/2)$ which shows that the angle α is uniquely determined by a_0 which in turn is uniquely determined by the spectrum.

By elementary geometry,

$$W = \frac{h}{\sin\alpha} \implies L = \frac{P}{2} - \frac{h}{\sin\alpha}, \quad A = h\left(\frac{P}{2} - \frac{h}{\sin\alpha}\right). \tag{3.1}$$

It is an exercise for the reader to show that the only solution of h consistent with the geometry is

$$h = \frac{P\sin\alpha}{4} - \sin\alpha\frac{\sqrt{\frac{P^2}{4} - \frac{4A}{\sin\alpha}}}{2}.$$

Consequently, the heat trace determines P, A, a_0, α, h, W, and L, which uniquely determines the parallelogram. ■

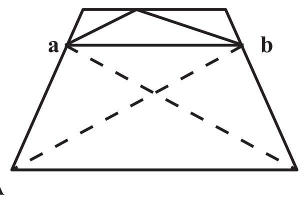

Figure 8. An (impossible) closed geodesic in an acute trapezoid.

Exercise 4. *Use the first two terms in* (2.7), *together with Hillairet's theorem [**20**], which states that the wave trace uniquely determines the length of the shortest closed geodesic, to prove that one can hear the shape of a parallelogram.*

Definition. An *acute trapezoid* (see Figure 7) is a convex quadrilateral which has two parallel sides of lengths b and B with $B \geq b$, and two nonparallel sides known as *legs* of lengths ℓ and ℓ'. The side of length B is the *base,* and the angles at the base α and β satisfy

$$\alpha + \beta < \frac{\pi}{2}. \tag{3.2}$$

Lemma 11. *The length of the shortest closed geodesic of an acute trapezoid is twice the height.*

Proof. The length of the shortest closed geodesic is realized by a path which meets either four, three, or two of the sides of the trapezoid. If a closed geodesic meets all four sides, then it is clearly longer than $2h$. By (3.2) there are precisely two parallel sides in the trapezoid, so by Snell's law the only closed geodesics which meet precisely two sides are those which go between the two parallel sides and have length $2h$. In the case of triangular closed geodesics, if two vertices are on the parallel sides, then the length exceeds $2h$. Since the angle of incidence is equal to the angle of reflection, when the closed geodesic meets a leg with an acute angle, it will be reflected down towards the base. If a closed geodesic meets a leg at an obtuse angle, this means that the geodesic came from the base, so in any case, the vertices of the triangle are on the legs and the base. By Snell's law the segment joining the vertex on each leg to the opposite vertex at the base meets the leg in a right angle; see Figure 8. The triangle aBA has one angle equal to $\frac{\pi}{2}$, one angle equal to the angle at A, and another angle less than or equal to the angle at B; by (3.2) the sum of these angles is less than π. The same holds for the triangle bAB. Consequently there are no such closed geodesics. ∎

By Theorem 2 in [**20**], the length of the shortest closed geodesic is one of the singularities of the wave trace, and the length of this geodesic is uniquely determined by the wave trace. So, $2h$ is a spectral invariant, and further spectral invariants can be extracted from the short time asymptotic expansion of the heat trace (2.7): the area A and the perimeter P. For a trapezoid with base B and opposite parallel side of length b these are

$$A = \frac{B+b}{2}h \implies B + b = \frac{2A}{h}, \qquad P = B + b + \ell + \ell',$$

where ℓ and ℓ' are the lengths of the legs. This shows that the spectrum uniquely determines h, $B + b$, and $\ell + \ell'$. The short time asymptotic expansion of the heat trace also gives information about the angles. Let α and β denote the interior angles at the base B, assuming without loss of generality that $\alpha \geq \beta$. Then

$$\ell = h \csc\alpha, \quad \ell' = h\csc\beta, \quad \ell + \ell' = h(\csc\alpha + \csc\beta),$$

so

$$\csc\alpha + \csc\beta = \frac{\ell + \ell'}{h}. \tag{3.3}$$

Since the angles are α, $\pi - \alpha$, β, and $\pi - \beta$, the constant term in (2.7) is

$$a_0 = \frac{\pi^2}{24}\left(\frac{1}{\alpha(\pi-\alpha)} + \frac{1}{\beta(\pi-\beta)}\right) - \frac{1}{12}. \tag{3.4}$$

In the following lemma, we will prove that (3.3) and (3.4) uniquely determine the angles α and β. This means that the spectrum uniquely determines the angles, the height, the area, and the perimeter which all together uniquely determine the acute trapezoid.

Lemma 12. *Let p and q be real numbers. Then the solution of the system of equations*

$$\begin{cases} \csc(\alpha) + \csc(\beta) = p \\ (\alpha(\pi-\alpha))^{-1} + (\beta(\pi-\beta))^{-1} = q \end{cases} \tag{3.5}$$

if it exists, must be unique for $0 < \beta \leq \alpha \leq \pi/2$.

Proof. First, we use the second equation to show that each α uniquely determines a β. Solving the second equation for β in terms of α and q leads to a quadratic equation in β whose only solution consistent with the geometry is

$$\beta = \beta(\alpha) = \frac{\pi}{2} - \sqrt{\frac{\pi^2}{4} + \frac{\alpha(\pi-\alpha)}{1 - q\alpha(\pi-\alpha)}}. \tag{3.6}$$

We can therefore prove the lemma if we prove that the function

$$g(\alpha) := \csc(\alpha) + \csc(\beta(\alpha)) \tag{3.7}$$

has unique solution α for any given p. Analyzing this function directly is problematic, due to the presence of the unknown constant q in the expression for β (3.6), so to avoid this we implicitly differentiate the second equation of (3.5) $\beta'(\alpha) = -\frac{((\alpha(\pi-\alpha))^{-1})'}{((\beta(\pi-\beta))^{-1})'}$ which shows that

$$g'(\alpha) = -\csc(\alpha)\cot(\alpha) + \csc(\beta(\alpha))\cot(\beta(\alpha)) \cdot \frac{((\alpha(\pi-\alpha))^{-1})'}{((\beta(\pi-\beta))^{-1})'}. \tag{3.8}$$

Since

$$\frac{g'(\alpha)}{((\alpha(\pi-\alpha))^{-1})'} = -\frac{\csc(\alpha)\cot(\alpha)}{((\alpha(\pi-\alpha))^{-1})'} + \frac{\csc(\beta(\alpha))\cot(\beta(\alpha))}{((\beta(\pi-\beta))^{-1})'},$$

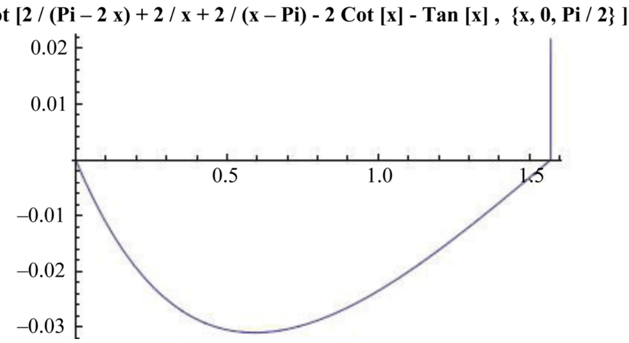

Figure 9. The graph of $(\log f)'$.

we see that

$$g'(\alpha) = -\frac{\pi - 2\alpha}{\alpha^2(\pi-\alpha)^2}\left(f(\alpha) - f(\beta)\right), \quad f(\alpha) := \frac{\alpha^2(\pi-\alpha)^2\cos\alpha}{\pi - 2\alpha\sin^2\alpha}. \tag{3.9}$$

Note that $f(\alpha) = \frac{\csc(\alpha)\cot(\alpha)}{((\alpha(\pi-\alpha))^{-1})'}$ is an equivalent expression.

It turns out that the logarithmic derivative of f is pleasantly simple

$$(\log(f(\alpha)))' = \frac{f'(\alpha)}{f(\alpha)} = \frac{2}{\pi - 2\alpha} + \frac{2}{\alpha} + \frac{2}{\alpha - \pi} - 2\cot(\alpha) - \tan(\alpha).$$

Since we assumed $\alpha \le \beta < \frac{\pi}{2}$ by (3.9), we see that if f is strictly monotone, then $g'(\alpha) \neq 0$ for all $\alpha \in (0, \beta)$. Since f is strictly monotone is implied by $\log(f(\alpha))' < 0$, the lemma is reduced to proving the following.

Claim: The function

$$u(\alpha) := \frac{f'(\alpha)}{f(\alpha)} < 0 \quad \forall \alpha \in \left(0, \frac{\pi}{2}\right). \tag{3.10}$$

A graph produced by Mathematica numerically proves the claim (see Figure 9). However, some readers may be interested to see that it is possible to prove this "by hand."

We compute

$$u''(\alpha) = \frac{16}{(\pi-2\alpha)^3} + \frac{4}{\alpha^3} + \frac{4}{(\alpha-\pi)^3} - \frac{4\cos(\alpha)}{\sin^3(\alpha)} - \frac{2\sin(\alpha)}{\cos^3(\alpha)}.$$

Using the trigonometric identities $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha)$, and $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha)$, it follows that

$$u''(\alpha) = \frac{4}{(\alpha-\pi)^3} + 4v(\alpha) + 2v\left(\frac{\pi}{2} - \alpha\right), \; v(\alpha) := \frac{1}{\alpha^3} - \frac{\cos(\alpha)}{\sin^3(\alpha)}. \tag{3.11}$$

We next compute

$$v'(\alpha) = -\frac{3}{\alpha^4} + \frac{1}{\sin^2(\alpha)} + \frac{3\cos^2(\alpha)}{\sin^4(\alpha)} = -\frac{3}{\alpha^4} + \frac{1}{\sin^2(\alpha)} + \frac{3(1-\sin^2(\alpha))}{\sin^4(\alpha)}$$

$$= -\frac{3}{\alpha^4} - \frac{2}{\sin^2(\alpha)} + \frac{3}{\sin^4(\alpha)} = \frac{-3\sin^4(\alpha) + 3\alpha^4 - 2\alpha^4 \sin^2(\alpha)}{\alpha^4 \sin^4(\alpha)}.$$

It follows that

$$v'(\alpha) \geq 0 \iff \left(\frac{\sin(\alpha)}{\alpha}\right)^4 \leq 1 - \frac{2\sin^2(\alpha)}{3}.$$

By the power series expansion for sine,

$$\left(\frac{\sin(\alpha)}{\alpha}\right)^4 \leq \left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^4 \implies 1 - \frac{2\sin^2(\alpha)}{3} \geq 1 - \frac{2}{3}\alpha^2 \left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^2.$$

To prove $v'(\alpha) \geq 0$ it suffices to show

$$\left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^4 \leq 1 - \frac{2}{3}\alpha^2 \left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^2.$$

This is none other than a quadratic equation in $t^2 := \left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^2$. By the quadratic formula, $t^4 + \frac{2}{3}\alpha^2 t^2 - 1 \leq 0$ if and only if

$$-\frac{\alpha^2}{3} - \sqrt{\frac{\alpha^4}{9} + 1} \leq t^2 \leq -\frac{\alpha^2}{3} + \sqrt{\frac{\alpha^4}{9} + 1}.$$

Substituting $(1 - \alpha^2/6 + \alpha^4/120)$ for t,

$$\left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^4 \leq 1 - \frac{2}{3}\alpha^2 \left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^2$$

$$\iff -\frac{\alpha^2}{3} - \sqrt{\frac{\alpha^4}{9} + 1} \leq \left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^2 \leq -\frac{\alpha^2}{3} + \sqrt{\frac{\alpha^4}{9} + 1}.$$

Clearly the left inequality always holds. Since $\left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^2 \leq 1 - \frac{\alpha^2}{3} + \frac{2\alpha^4}{45}$, and $\left(1 + \frac{2\alpha^4}{45}\right)^2 \leq 1 + \frac{\alpha^4}{9} \implies 1 + \frac{2\alpha^4}{45} \leq \sqrt{\frac{\alpha^4}{9} + 1}$, we have

$$\left(1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120}\right)^2 \leq -\frac{\alpha^2}{3} + 1 + \frac{2\alpha^4}{45} \leq -\frac{\alpha^2}{3} + \sqrt{\frac{\alpha^4}{9} + 1}.$$

This shows that $v'(\alpha) \geq 0$, which we can use to estimate $u''(\alpha)$. Since

$$u''(\alpha) = \frac{4}{(\alpha - \pi)^3} + 4v(\alpha) + 2v\left(\frac{\pi}{2} - \alpha\right),$$

and v is increasing and therefore $v(\pi/2 - \alpha)$ is decreasing, on $(0, \pi/4]$ we have

$$u''(\alpha) \geq \frac{4}{(\pi/4 - \pi)^3} + 4v(0) + 2v(\pi/4) = -\frac{256}{27\pi^3} + \frac{128}{\pi^3} - 2 > 0.$$

Note that although perhaps not immediately obvious, it is nonetheless straightforward to compute using the Taylor series expansion for sine and cosine that $v(0) = 0$. Similarly, on $(\pi/4, \pi/2]$ we have

$$u''(\alpha) \geq -\frac{4}{(\pi/2)^3} + 4v(\pi/4) + 2v(0) = \frac{224}{\pi^3} - 4 > 0.$$

Consequently u is convex. Since $u(0) = u(\pi/2) = 0$, $u(\alpha) < 0$ on $(0, \pi/2)$. ■

Remark. Durso proved that isospectral triangles are congruent using both the heat and the wave traces [**12**] requiring over fifty pages, whereas Grieser and Maronna gave a significantly shorter and more elementary proof using only the heat trace in [**19**]. The following exercise shows that the first three coefficients in the short time asymptotic expansion of the heat trace do not suffice to prove our result for trapezoids. Therefore, to prove our result for trapezoids, further information, for example from the wave trace, is necessary.

Exercise 5. *Prove that there exist trapezoids with identical area, perimeter, and constant term a_0 in the heat trace which are not congruent.*

4. HEARING THE REGULAR n-GON

Proof of Theorem 4. We begin with a short proof of Theorem 4 based on Proposition 6.

Proof of Theorem 4. We assume that for a fixed $n \geq 3$ there is an n-gon Q such that

$$\lambda_k(Q) = \lambda_k(R_n), \quad \forall k \geq 1,$$

where R_n is a regular n-gon. In § 2 we proved that the function

$$f(Q) = \frac{\text{Area of } Q}{(\text{Perimeter of } Q)^2}$$

is uniquely maximized among all n-gons by a regular n-gon. Since the spectrum determines the heat trace, by (2.7) it follows that Q and R_n have the same area and perimeter so,

$$f(Q) = f(R_n).$$

Consequently, Q is also regular, and since Q and R_n have the same perimeter and area, they are the same regular n-gon up to a rigid motion. ■

Proof of the weak Pólya–Szegő theorem. The proof of Theorem 5 is based on Proposition 7, Theorem 4, and Proposition 10.

Proof of Theorem 5. For the sake of contradiction, we assume that there exist convex n-gons $\{\Omega_k\}_{k=1}^{\infty}$ with

$$\lambda_i(\Omega_k) = \lambda_i(P_k), \quad \forall i < k, \tag{4.1}$$

where P_k is a regular n-gon, and each Ω_k is *not* regular. The scaling property for the eigenvalues shows that (4.1) is equivalent to

$$\lambda_i(\tilde{\Omega}_k) = \lambda_i(P), \quad \forall i < k, \tag{4.2}$$

where P is the regular n-gon with diameter equal to 1, and $\tilde{\Omega}_k$ is Ω_k scaled by d_k^{-1}, where d_k is the diameter of P_k. Slightly abusing notation, we shall write Ω_k rather than $\tilde{\Omega}_k$.

By Proposition 7, since for $k > 3$ the domains Ω_k have the same first two eigenvalues, it follows that Ω_k can neither collapse nor explode as $k \to \infty$. Passing to a subsequence if necessary, we assume that

$$\Omega_k \to \Omega_0,$$

where Ω_0 is an m-gon for $m \le n$. By the continuity of the eigenvalues under continuous deformation of a domain, for each j, $\lambda_j(P) = \lambda_j(\Omega_k) \to \lambda_j(\Omega_0)$ as $k \to \infty$. Consequently Ω_0 is isospectral to P, so by the same argument used in Proposition 6, we conclude that Ω_0 is an n-gon, and in fact $\Omega_0 \cong P$ by Theorem 4. By Proposition 10, Ω_k are parametrized by $\{\varepsilon_k\}_{k=0}^{\infty} \subset \mathbb{R}^{2n-3}$, and since $\Omega_k \to \Omega_0$ as $k \to \infty$, we correspondingly have $\varepsilon_k \to \varepsilon_0$ as $k \to \infty$. It follows that the set $\{\varepsilon_k\}_{k=0}^{\infty}$ is compact in $\mathbb{R}^{2n-3}$. By Theorem 9 there exists $N = N(n)$ such that if the first N eigenvalues of Ω_k are equal to the first N eigenvalues of Ω_0, then all eigenvalues are equal. Consequently, for $k > N$, Ω_k and Ω_0 are isospectral. By Theorem 4, $\Omega_k \cong \Omega_0$ for all $k > N$ which is a contradiction since Ω_k is not regular for all $k \in \mathbb{N}$. ■

5. CONJECTURES AND OPEN PROBLEMS. The Pólya–Szegő conjecture would indicate that $N = N(n)$ in Theorem 5 may be taken equal to 1, however, that conjecture assumes the polygons all have fixed area. Since Theorem 5 holds without the area-normalization assumption, we propose that the original conjecture may be strengthened as follows.

Conjecture 2 (Strong Pólya–Szegő Conjecture). *The number $N = N(n)$ in Theorem 5 may be taken equal to* 2.

Exercise 6. *Prove that the first eigenvalue is insufficient to uniquely distinguish a regular n-gon amongst all convex n-gons without further assumptions.*

For the special case of triangles, Antunes and Freitas made the natural conjecture in [**2**], that the first three eigenvalues uniquely determine a triangle. They provided a vast amount of supporting numerical data, so that one may consider the conjecture to be "numerically" a theorem. Since three parameters uniquely determine a triangle, this conjecture may seem rather obvious at first glance, and one may ask more generally whether any three eigenvalues would suffice. Intriguingly, [**2**] showed that the first, second, and fourth eigenvalues *cannot* uniquely determine a triangle, so the conjecture appears to be more subtle than one might expect.

Conjecture 3 (Antunes–Freitas). *If the first three eigenvalues of two triangles coincide, then they are identical up to a rigid motion.*

One can also consider isospectral problems for the *length spectrum,* the set of lengths of closed geodesics. A natural way to make a pure mathematical conjecture is to base the conjecture on observations in physics or nature. In this setting we are reminded of bats who use echolation to determine their location relative to objects and prey. A bat emits a sound (which is generally inaudible to the human ear) and remarks the time(s) at which the sound is reflected back. If the bat were in a convex environment, one can imagine that the boundary is like the boundary of a domain in $\mathbb{R}^3$, and the return times of echolation are similar to the lengths of closed geodesics. It is only possible for the bat to detect a finite amount of return times, which mathematically correspond to the lengths of finitely many closed geodesics. Inspired by nature, we make the following "bat conjecture."

Conjecture 4. *For each $n \geq 3$ there exists $N = N(n)$ such that if the lengths of N primitive closed geodesics of two convex n-gons coincide, then they are identical up to a rigid motion.*

Finally two natural questions arise from our work, the more tractable of which is the following.

Question 13. *Is a trapezoid uniquely determined by its spectrum? Do there exist isospectral trapezoids which are not isometric?*

More generally, we are very curious to know the answer to the following.

Question 14. *Can one hear the shape of a convex 4-sided drum?*

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